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MATHEMATICS

ON THE MONOTONICITY AND SUBADDITIVITY OF NORMING PERTURBATIONS FOR BANACH FUNCTION SPACES

BY

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W. A. J. LUXEMBURG and A. C. ZAANEN in [3, Note IV, p. 262] posed the question of whether the difference between a function norm ϱ and its Lorentz norm ϱ_L ever fails to be a function seminorm. Indeed, they further asked whether it could fail to be monotone.

In Section 2 of this paper, we present an example (Ex. 2.3) of a sequence space with a function norm ϱ such that the difference $\varrho-\varrho_L$ fails to be either monotone or subadditive. Thus, the questions posed in [3] are answered in the affirmative.

We introduce in Section 1 some terminology which is efficacious to the statement of our results in Section 3 where we study the monotonicity and subadditivity of the difference $\varrho-\varrho_L$. The main result of this section is Theorem 3.9, which says, in particular, that the difference $\varrho-\varrho_L$ is monotone if and only if it is subadditive (in a certain sense, see Section 3).

Throughout this paper, we will assume that the reader has a familiarity with the material in [3, Notes I–V].

1. NOTATION AND TERMINOLOGY

Let $(X, \mathscr{L}, \mu)$ be a σ-finite measure space. In keeping with [3] (except in that we restrict ourselves to the real number situation), we will denote by M the set of equivalence classes of extended real valued μ-measurable functions defined on X, and by M^+ the set of equivalence classes of extended positive real valued μ-measurable functions defined on X.

DEFINITION 1.1. *Let ϱ be a Fatou, saturated function norm. The mapping τ of M^+ into the extended real numbers is called a ϱ-norming perturbation if τ satisfies the following conditions:*

(i) $0 \leqslant \tau(u) \leqslant \infty$ *for all* $u \in M^+$,

(ii) *the function* $(\varrho+\tau)$, *defined by* $(\varrho+\tau)(u)=\varrho(u)+\tau(u)$, *is a saturated function norm,*

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(iii) *if* $(\varrho+\tau)_L$ *is the Lorentz norm associated with* $(\varrho+\tau)$ (*i.e.*, $(\varrho+\tau)_L(u)=$
$= \min\{\lim_n (\varrho+\tau)(u_n)\colon 0\leqslant u_n \uparrow u\}$, $u\in M^+$), *then* $(\varrho+\tau)_L=\varrho$.

(iv) *if* $\varrho(u)=\infty$, *then* $\tau(u)=+\infty$.

Remark 1.2. For the sake of brevity, we will restrict our attention throughout to Fatou, saturated function norms ϱ. All of our results can be extended, with due care, to Fatou, function seminorms ϱ by defining τ to be a *ϱ-seminorming perturbation* if it satisfies:

(i)′ $0\leqslant\tau(u)\leqslant\infty$ for $u\in M^+$,

(ii)′ the mapping $(\varrho+\tau)$, defined by $(\varrho+\tau)(u)=\varrho(u)+\tau(u)$, is a function seminorm,

(iii)′ if $(\varrho+\tau)_L$ is the Lorentz seminorm associated with $(\varrho+\tau)$, then $(\varrho+\tau)_L=\varrho$,

(iv)′ if $\varrho(u)=\infty$, then $\tau(u)=\infty$.

Let τ be a ϱ-norming perturbation. By $L_{\varrho,\tau}$ we will denote the set of all $u\in M$ such that $(\varrho+\tau)(|u|)<\infty$. It is easy to see that $L_{\varrho,\tau}=\{u\in L_\varrho\colon \tau(|u|)<\infty\}$.

Remark 1.3. It follows directly from condition (ii) of Definition 1.1 that, for $u, v\in M^+$, the following conditions hold:

(i) If $u=0$, then $\tau(u)=0$,

(ii) $\tau(au)=a\tau(u)$ for every finite constant $a\geqslant 0$,

(iii) If $\tau(u)=\infty$ and $v\geqslant u$, then $\tau(v)=\infty$ whenever $\varrho(v)<\infty$.

By virtue of condition (iv) of Definition 1.1 and condition (iii) of Remark 1.3, it follows that a ϱ-norming perturbation τ is a function seminorm if and only if it is monotone and subadditive on $L^+_{\varrho,\tau}$.

From condition (iii) of Definition 1.1, it follows that, for $u\in L_\varrho$, $(\varrho+\tau)(u)-(\varrho+\tau)_L(u)=(\varrho+\tau)(u)-\varrho(u)=\tau(u)$.

Remark 1.4. It was essentially proved in [3, Note IV] that, for any saturated function norm ϱ, the difference $\varrho-\varrho_L$ is (on $L^+_{\varrho_L}$) the same as a ϱ_L-norming perturbation restricted to $L^+_{\varrho_L}$. In fact, if we take

$$\tau(u)=\begin{cases}\varrho(u)-\varrho_L(u) & \text{for } u\in L^+_{\varrho_L},\\ \infty & \text{whenever } \varrho_L(u)=\infty \text{ and } u\in M^+,\end{cases}$$

then τ will actually be a ϱ_L-norming perturbation.

In a figurative sense, therefore, the study of ϱ-norming perturbations is equivalent to the study of the "difference" between the theory of saturated function norms and the theory of Fatou, saturated function norms (the theory of Fatou, saturated function norms being itself equivalent to the theory of Köthe-Toeplitz spaces introduced by G. G. Lorentz and D. G. Wertheim in [1] (see also [2])).

We find it convenient to use the concept of a semi-continuous norm, the definition of which follows.

A norm $\|\cdot\|$ defined on a vector lattice L under which L becomes a normed vector lattice is said to be *semi-continuous* (*σ-semi-continuous*) if for every net $\{x_\alpha\} \subset L^+$ (sequence $\{x_n\} \subset L^+$) with $x_\alpha \uparrow x_0 \in L^+$ ($x_n \uparrow x_0 \in L^+$) we have that $\sup \|x_\alpha\| = \|x_0\|$ ($\sup \|x_n\| = \|x_0\|$).

A Riesz space L is said to be *order separable* if whenever K is a subset of L which has a supremum, then there is a countable subset K' of K such that $\sup K' = \sup K$ (equivalently, every net $\{x_\alpha\} \subset L^+$ with $x_\alpha \uparrow x_0 \in L^+$ contains a sequence $\{x_n\}$ such that $x_n \uparrow x_0$).

REMARK 1.5. We will need to know that if ϱ is a saturated function norm, then L_ϱ is order separable. To see this, consider first the associated norms ϱ' and ϱ'' (see [3, Def. 9.1 in Note IV]). By combining several of the results in [3, Note IV], we have that both ϱ' and ϱ'' are Fatou, saturated function norms and that $\varrho'' = (\varrho')'$. Combining this with [3, Th. 5.3 in Note II] we have that $L_{\varrho'}$ is complete. Hence, by virtue of [3, Th. 12.3 (iii) in Note V], it follows that L_ϱ has a weak unit $e > 0$ in the sense of [3, Def. 12.2 (ii) in Note V]. Then the functional $E(f) = \int fe d\mu$ for $f \in L_\varrho$ is a strictly positive linear functional on L_ϱ. That L_ϱ is order separable now follows by application of [3, Th. 31.11 (i) in Note X].

REMARK 1.6. If the Riesz space L is order separable, then every σ-semi-continuous norm on L is semi-continuous. It therefore follows from Remark 1.5 that every Fatou, saturated function norm ϱ when considered as a norm on L_ϱ is semi-continuous.

2. THE EXAMPLE

Before we can present the example, we must first state a known result concerning norm extensions from a vector lattice to its Dedekind completion. We must also establish one preliminary lemma.

Let $(L, \|\cdot\|)$ be a normed vector lattice with norm $\|\cdot\|$. Furthermore, let $L^\wedge$ be the Dedekind completion of L and consider the following mapping of $L^\wedge$ into the positive reals:

$$\|u\|^* = \inf (\|v\| : v \geqslant |u| ; v \in L).$$

It is known that $\|\cdot\|^*$ is always a norm extending $\|\cdot\|$ from L to $L^\wedge$ (c.f., [6, p. 179]).

We will say that a subset K of a Riesz space L has *property D* (*property σ-D*) if for every $0 \leqslant u \in L$ there is a directed upward net u_τ (sequence u_n) of positive elements in K such that $u = \sup u_\tau$ ($u = \sup u_n$).

The lemma which we need is the following:

LEMMA 2.1. *Let ϱ_1 and ϱ_2 be two saturated function norms such that $L = L_{\varrho_1} = L_{\varrho_2}$ and such that $\varrho_1 = \varrho_2$ on a Riesz subspace $L' \subseteq L$ with property D. Assume in addition that ϱ_1 and ϱ_2 are semi-continuous on L'. Then* $\varrho_{1L} = \varrho_{2L}$.

PROOF. We note first that if $u \in M^+$ and $\{u_n\}$ is a sequence in L with $0 \leqslant u_n \uparrow u$, then there exists a sequence $\{v_n\}$ in L' with $0 \leqslant v_n \uparrow u$ such that $v_n \leqslant u_n$ for $n=1, 2, \ldots$. To see this, we note that since L is order separable (see Remark 1.5), we actually have that L' has property σ-D with respect to L. Thus, for each n, there exists a sequence $0 \leqslant v_{n,m} \uparrow_m u_n$ with $\{v_{n,m}\} \subset L'$ for $m, n=1, 2, \ldots$. Taking $z_n = \sup (v_{i,n};\ 1 \leqslant i \leqslant n)$, it is easy to check that $\{z_n\} \subset L'$, $0 \leqslant z_n \uparrow u$, and that $z_n \leqslant u_n$ for $n=1, 2, 3, \ldots$.

Now, it follows from the fact that ϱ_1 and ϱ_2 are saturated, and from what we have just seen, that, for every $u \in M^+$, there exist sequences $\{v_n^1\} \subset L'$ and $\{v_n^2\} \subset L'$ with $0 \leqslant v_n^i \uparrow_n u$ for $i=1, 2$ such that

$$\varrho_{1L}(u) = \lim_n \varrho_1(v_n^1) \text{ and } \varrho_{2L}(u) = \lim_n \varrho_2(v_n^2).$$

Our assertion will follow, therefore, if we can show that

$$\lim_n \varrho_1(v_n^1) = \lim_n \varrho_1(v_n^2) = \lim_n \varrho_2(v_n^2)$$

(where the last equality is by assumption). To this end, let n_0 be a fixed natural number. Then $v_{n_0}^1 \wedge v_n^2 \uparrow v_{n_0}^1 \wedge u = v_{n_0}^1$. Since ϱ_1 is semi-continuous on L', it follows that $\varrho_1(v_{n_0}^1 \wedge v_n^2) \uparrow \varrho_1(v_{n_0}^1)$. But, since $v_n^2 \geqslant v_{n0}^1 \wedge v_n^2$ for $n=1, 2, \ldots$, we have that $\lim_n \varrho_1(v_n^2) \geqslant \varrho_1(v_{n_0}^1)$ and hence that

$$\lim_n \varrho_1(v_n^2) \geqslant \lim_n \varrho_1(v_n^1).$$

Interchanging the roles of v_n^1 and v_n^2 above and applying the same argument, we obtain that $\lim_n \varrho_1(v_n^1) \geqslant \lim_n \varrho_1(v_n^2)$. The lemma is proved.

REMARK 2.2. With proper modification, the above lemma has an analogue in the more general setting of Archimedean, order separable Riesz spaces with the Egoroff property. It was shown in [4, p. 52] that whenever L is order separable and has the Egoroff property, then $\|\cdot\|_{1L} = \|\cdot\|_{2L}$ where $\|\cdot\|_1$ and $\|\cdot\|_2$ are any two norm extensions of $\|\cdot\|$ to $L^\wedge$ and are not necessarily semi-continuous.

In fact, most of our results will have analogues in this more general setting. We will not, however, go into this here.

We are now ready to present our example.

EXAMPLE 2.3. Let $L=c$ be the Riesz space of convergent sequences. Then $L^\wedge = l_\infty$; i.e., the Dedekind completion of L is the Riesz space of all bounded sequences. Consider the element $\hat{e} \in L^\wedge$, where $\hat{e} = (1, 2, 1, 2, \ldots)$.

Let ϱ be the function norm defined by

$$\varrho(u) = \inf (\lambda \colon \lambda\hat{e} \geqslant u,\ \lambda \text{ a real number})$$

for $u \in M^+$, where M is the space of all real sequences. Then $L_\varrho = L^\wedge$. It is easy to see that ϱ is a Fatou, saturated function norm.

Now, let $p = \varrho/L$ (the restriction of ϱ to L). Taking

$$p'(u) = \begin{cases} p^*(u) & \text{for } u \in (L^\wedge)^+, \\ \infty & \text{for } u \in M^+ - L^\wedge, \end{cases}$$

we have, by virtue of Lemma 2.1 (since L has property D with respect to $L^{\wedge}$, $p'=\varrho$ on L, and ϱ is semi-continuous on L), that $p'_L=\varrho_L=\varrho$.

Letting

$$\tau(u)=\begin{cases} p'(u)-\varrho(u) \text{ for } u\in(L^{\wedge})^+, \\ \infty \text{ for } u\in M^+-(L^{\wedge})^+, \end{cases}$$

we have that τ is a ϱ-norming perturbation.

We note the following two things:

(1) τ is not monotone. To see this, consider the element $\hat{e}=(1, 2, 1, 2, \ldots)$. We have that $\varrho(\hat{e})=1$ and $p'(\hat{e})=2$. So, $\tau(\hat{e})=p'(\hat{e})-\varrho(\hat{e})=1$. On the other hand, if $e=(2, 2, \ldots)$, then $e\geqslant\hat{e}$ and $p'(e)=p(e)=\varrho(e)$ (since $e\in L$). So, $\tau(e)=\varrho(e)-\varrho(e)=0$. This establishes that τ is not monotone.

(2) τ is not subadditive. To see this, let $x_1=(1, 1, 1, \ldots)$ and $x_2=(0, 1, 0, 1, \ldots)$. Then

$$\tau(x_1)=0,\ \ \tau(x_2)=p'(x_2)-\varrho(x_2)=1-\tfrac{1}{2}=\tfrac{1}{2}$$

and

$$\tau(x_1+x_2)=\tau(\hat{e})=1.$$

So, $\tau(x_1+x_2)>\tau(x_1)+\tau(x_2)$. This establishes that τ is not subadditive.

3. Monotonicity and subadditivity

In what follows, when we say that the ϱ-norming perturbation τ is *subadditive on* $L_{\varrho,\tau}$, we will mean that, for every $u, v\in L_{\varrho,\tau}$, we have $\tau(u+v)\leqslant\tau(u)+\tau(v)$, where $\tau(u+v)=\tau(|u+v|)$, $\tau(u)=\tau(|u|)$ and $\tau(v)=\tau(|v|)$.

The next result establishes half of the relationship between the subadditivity of a ϱ-norming perturbation and its monotonicity.

Lemma 3.1. *Suppose that the ϱ-norming perturbation τ is subadditive on $L_{\varrho,\tau}$. Then τ is monotone.*

Proof. We saw in Section 1 that we need only concern ourselves with the elements in $L^+_{\varrho,\tau}$.

We first note that, since $(\varrho+\tau)_L(u)=\varrho(u)$ for every $u\in L^+_{\varrho,\tau}$, there exists $0\leqslant u_n\uparrow u$ such that $\tau(u_n)\to 0$ for $n=1, 2, \ldots$. Now, let $0<u<v$ with $u, v\in L_{\varrho,\tau}$ and let the sequence $\{v_n\}$ in $L_{\varrho,\tau}$, with $0\leqslant v_n\uparrow(v-u)$, be such that $\tau(v_n)\to 0$. It follows from the monotonicity of $(\varrho+\tau)$ that $\varrho(u+v_n)-\varrho(v)\leqslant\tau(v)-\tau(u+v_n)$.

Let $\varepsilon>0$ be chosen. Choose $n=n_1$ so that $\tau(v_n)\leqslant\varepsilon/2$ for $n\geqslant n_1$. Choose $n=n_2$ so that $|\varrho(u+v_n)-\varrho(v)|\leqslant\varepsilon/2$ for $n\geqslant n_2$ (that such can be done follows from the semi-continuity of ϱ).

Taking $N=\max(n_1, n_2)$ we have that, for $n\geqslant N$,

$$-\varepsilon/2\leqslant\tau(v)-\tau(u+v_n)\leqslant\tau(v)-\tau(u)+\varepsilon/2$$

(where the last inequality follows from the subadditivity on $L_{\varrho,\tau}$ of τ). But then $-\varepsilon\leqslant\tau(v)-\tau(u)$. Since the choice of $\varepsilon>0$ was arbitrary, we have established that $\tau(v)-\tau(u)\geqslant 0$. The lemma is proved.

The result that the monotonicity of a ϱ-norming perturbation implies that it is subadditive involves a little more work.

A subset K of a Riesz space L is said to be *solid* if whenever $x, y \in L$ with $|x| \leqslant |y|$ and $y \in K$ we have $x \in K$.

REMARK 3.2. If τ is a monotone ϱ-norming perturbation, then $A = \{u \in L_{\varrho,\tau} : \tau(|u|) = 0\}$ is a solid subset of $L_{\varrho,\tau}$ with property σ-D.

W. A. J. LUXEMBURG and A. C. ZAANEN have shown [3, Note IV, Th. 11.10, p. 262] that, in fact, the following is true:

If τ is a monotone ϱ-norming perturbation, then there exists a sequence $X_n \uparrow X$ such that $\tau(u) = 0$ for every $0 \leqslant u \in L_{\varrho,\tau}$ such that u is bounded and vanishing outside some X_n (where n depends on u). (We have paraphrased the result in [3] in order to be consistent with our terminology.) Throughout the rest of this paper we will refer to this property as (*).

Thus, there is actually an ideal in $L_{\varrho,\tau}$ with property σ-D on which the ϱ-norming perturbation τ vanishes.

We saw in Example 2.3 that we can have a ϱ-norming perturbation such that even (*) holds but τ fails to be monotone (τ vanishes on the finitely non-zero sequences).

The following is a characterization of monotonicity.

LEMMA 3.3. *Let τ be a ϱ-norming perturbation. Then the following two conditions are equivalent*:

(i) *τ is monotone*;

(ii) *There exists a solid subset A of $L_{\varrho,\tau}$ with property σ-D such that if $u, v \in L^+_{\varrho,\tau}$ with $v \in A$, then $\tau(u+v) \geqslant \tau(u)$.*

PROOF. That the monotonicity of τ implies condition (ii) is obvious. We prove, therefore, only that condition (ii) implies that τ is monotone. To this end, let $u, v \in L_{\varrho,\tau}$ with $0 \leqslant u < v$. Let A be a subset of $L_{\varrho,\tau}$ with the properties of condition (ii). Since A has property σ-D, there exists a sequence $0 \leqslant v_n \uparrow (v-u)$ with $\{v_n\} \subset A$. Since

$$\varrho(u+v_n) + \tau(u+v_n) \leqslant \varrho(v) + \tau(v) \text{ for } n = 1, 2, \ldots,$$

we have that, for all n,

$$\tau(u) - \tau(v) \leqslant \tau(u+v_n) - \tau(v) \leqslant \varrho(v) - \varrho(u+v_n).$$

Since ϱ is semi-continuous, we have that

$$\varrho(v) - \varrho(u+v_n) \to \varrho(v) - \varrho(v) = 0,$$

and therefore

$$\tau(u) - \tau(v) \leqslant 0.$$

The lemma is proved.

REMARK 3.4. In Example 2.3, it is not difficult to see that every solid subset A of $L^\wedge$ with property σ-D must contain the finitely non-zero sequences. Therefore, to see that condition (ii) of Lemma 3.3 is not satisfied one need only consider, for instance, the elements $\hat{e}=(1, 2, 1, 2, \ldots)$ and $x=(1, 0, 0, \ldots)$.

LEMMA 3.5. *Let τ be a monotone ϱ-norming perturbation. Let $0 \leqslant e \in L_{\varrho,\tau}$ and let $E=\{x \in X: e(x)>0\}$. Then there exist measurable $E_n \uparrow E$ such that $(1/n)\chi_{E_n} \leqslant e\chi_{E_n} \leqslant n\chi_{E_n}$ for $n=1, 2, \ldots$, and such that for each n we have $\tau(\chi_{E_n})=0$.*

PROOF. We remark first that for any subset $Z \in \mathscr{L}$, if we have sequences $Y_n \uparrow Z$ and $D_n \uparrow Z$ with $Y_n, D_n \in \mathscr{L}$ for $n=1, 2, \ldots$, then $D_n \cap Y_n \uparrow Z$. The proof of this is elementary.

Now, let $D_n \uparrow X$ be the sequence guaranteed by (*). Let $B_n=\{x \in E: e(x) \geqslant (1/n)\}$ and $C_n=\{x \in E: e(x) \leqslant n\}$ for $n=1, 2, \ldots$. Then $B_n \uparrow E$ and $C_n \uparrow E$, so

$$A_n = B_n \cap C_n \uparrow E.$$

Likewise, since $D_n \cap E \uparrow E$, we have that

$$(D_n \cap E) \cap A_n = D_n \cap A_n \uparrow E.$$

The desired sequence $\{E_n\}$ is obtained by taking $E_n = A_n \cap D_n$ for $n=1, 2, \ldots$. The lemma is proved.

LEMMA 3.6. *Let τ be a monotone ϱ-norming perturbation and let e_1, e_2 be positive elements of $L_{\varrho,\tau}$ such that $\{x: e_1(x)>0\}=\{x: e_2(x)>0\}=E$. Then there exists a sequence $\{E_n\}$ of measurable subsets of E with $E_n \uparrow E$ such that*

$$(1/n^3)\chi_{E_n} \leqslant (1/n^2)e_1\chi_{E_n} \leqslant e_2\chi_{E_n} \leqslant n^2 e_1\chi_{E_n} \leqslant n^3\chi_{E_n}$$

for $n=1, 2, \ldots$, and such that, for each n, $\tau(\chi_{E_n})=0$.

PROOF. Let $E_{1,n}$ (respectively, $E_{2,n}$) be the sequence obtained for e_1 (respectively, e_2) by application of Lemma 3.5. Taking $E_n = E_{1,n} \cap E_{2,n}$ for $n=1, 2, \ldots$, we have that $E_n \uparrow E$, $\tau(\chi_{E_n})=0$, $(1/n)\chi_{E_n} \leqslant e_1\chi_{E_n} \leqslant n\chi_{E_n}$, and $(1/n)\chi_{E_n} \leqslant e_2\chi_{E_n} \leqslant n\chi_{E_n}$ for each n. The result now follows easily.

LEMMA 3.7. *Let τ be a monotone ϱ-norming perturbation. Let e_1, e_2 be positive elements of $L_{\varrho,\tau}$ such that*

$$\{x \in X: e_1(x)>0\}=\{x \in X: e_2(x)>0\}=E.$$

Then $\tau(e_1+e_2) \leqslant \tau(e_1)+\tau(e_2)$.

PROOF. Let $E_n \uparrow E$ be the sequence obtained by application of Lemma 3.6. We claim first that

$$\tau(e_1+e_2\chi_{E_n})=\tau(e_1), \tag{1}$$

$$\tau(e_2+e_1\chi_{E_n})=\tau(e_2) \text{ for each } n. \tag{2}$$

We will show only (1) since the proof of (2) is analogous. Since τ is monotone, we need only show that $\tau(e_1+e_2\chi_{E_n})\leqslant\tau(e_1)$. It follows also from the monotonicity of τ that, for a fixed positive integer m, we have $\tau(e_1+e_2\chi_{E_m})\leqslant\tau(e_1+m^2e_1\chi_{E_n})$ for $n\geqslant m$. Hence, since $(\varrho+\tau)$ is a function norm, we have that

$$\varrho(e_1+m^2e_1\chi_{E_n})-\varrho(e_1)-m^2\varrho(e_1\chi_{E_n})\leqslant$$
$$\tau(e_1)-\tau(e_1+m^2e_1\chi_{E_n})\leqslant\tau(e_1)-\tau(e_1+e_2\chi_{E_m}).$$

Since ϱ is semi-continuous, it follows that

$$\varrho(e_1+m^2e_1\chi_{E_n})-\varrho(e_1)-m^2\varrho(e_1\chi_{E_n})\to 0 \text{ as } n\to\infty.$$

Hence, $\tau(e_1)-\tau(e_1+e_2\chi_{E_m})\geqslant 0$. The equality (1) is proved.

Now, by the subadditivity of $(\varrho+\tau)$, the monotonicity of τ, and equalities (1) and (2) above, we have that

$$\varrho(e_1+e_2\chi_{E_n}+e_1\chi_{E_n}+e_2)-\varrho(e_1+e_2\chi_{E_n})-\varrho(e_2+e_1\chi_{E_n})\leqslant$$
$$\tau(e_1+e_2\chi_{E_n})+\tau(e_2+e_1\chi_{E_n})-\tau(e_1+e_2)=$$
$$\tau(e_1)+\tau(e_2)-\tau(e_1+e_2).$$

Applying once more the semi-continuity of ϱ, we see that $\tau(e_1)+\tau(e_2)-$ $-\tau(e_1+e_2)\geqslant 0$. The assertion is proved.

LEMMA 3.8. *Let τ be a monotone ϱ-norming perturbation. Then τ is subadditive.*

PROOF. Let $e_1, e_2\in L^+_{\varrho,\tau}$ and $\varepsilon>0$ be chosen. For some positive integer n_1 we have that

(**) $\qquad \tau(e_1)\leqslant\tau(e_1+(1/n)e_2)\leqslant\tau(e_1)+\varepsilon/2$ for all integers $n\geqslant n_1$.

Indeed, we have

$$\varrho(e_1+(1/n)e_2)+\tau(e_1+(1/n)e_2)\leqslant\varrho(e_1)+\varrho(e_2/n)+\tau(e_1)+(1/n)\tau(e_2),$$

and so

$$\tau(e_1)\leqslant\tau(e_1+(1/n)e_2)\leqslant\varrho(e_1)+\varrho((1/n)e_2)-\varrho(e_1+(1/n)e_2)+\tau(e_1)+(1/n)\tau(e_2).$$

Noting that $\varrho(e_1+(1/n)e_2)\geqslant\varrho(e_1)$, we have that

$$\tau(e_1)\leqslant\tau(e_1+(1/n)e_2)\leqslant(1/n)\varrho(e_2)+\tau(e_1)+(1/n)\tau(e_2).$$

To establish (**), we need only choose a natural number n_1 large enough so that $(1/n)(\varrho(e_2)+\tau(e_2))\leqslant\varepsilon/2$ whenever $n\geqslant n_1$.

Similarly, it can be shown that for some natural number n_2 we have that

$$\tau(e_2)\leqslant\tau(e_2+(1/n)e_1)\leqslant\tau(e_2)+\varepsilon/2 \text{ for all integers } n\geqslant n_2.$$

It follows from Lemma 3.7 that, for all natural numbers n,

$$\tau(e_1+(1/n)e_2+e_2+(1/n)e_1)\leqslant\tau(e_1+(1/n)e_2)+\tau(e_2+(1/n)e_1).$$

Hence, for $n \geqslant \max(n_1, n_2)$, we have that

$$\begin{aligned} 0 &\leqslant \tau(e_1+(1/n)e_2)+\tau(e_2+(1/n)e_1)-\tau(e_1+e_2+(1/n)(e_1+e_2)) \\ &\leqslant \tau(e_1+(1/n)e_2)+\tau(e_2+(1/n)e_1)-\tau(e_1+e_2) \\ &\leqslant \tau(e_1)+\varepsilon/2+\tau(e_2)+\varepsilon/2-\tau(e_1+e_2). \end{aligned}$$

So,

$$-\varepsilon \leqslant \tau(e_1)+\tau(e_2)-\tau(e_1+e_2).$$

Since the choice of $\varepsilon > 0$ was arbitrary, this establishes that $0 \leqslant \tau(e_1) + \tau(e_2) - \tau(e_1+e_2)$. Thus the proof is complete.

The reader should note that if the ϱ-norming perturbation τ is subadditive and monotone, then it is subadditive on $L_{\varrho,\tau}$.

The following theorem summarizes the results of this section.

THEOREM 3.9. *Let τ be a ϱ-norming perturbation. The following are equivalent:*

(i) *τ is a function seminorm,*

(ii) *τ is monotone,*

(iii) *there exists a solid subset A of $L_{\varrho,\tau}$ with property σ-D such that if $u, v \in L^+_{\varrho,\tau}$ with $v \in A$, then $\tau(u+v) \geqslant \tau(u)$,*

(iv) *τ is subadditive on $L_{\varrho,\tau}$.*

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REFERENCES

1. LORENTZ, G. G. and D. G. WERTHEIM, Representation of linear functionals on Köthe spaces, Can. J. Math., 5, 568–575 (1953).
2. LUXEMBURG, W. A. J., Banach Function Spaces, Thesis, Delft Inst. of Techn., Assen (Netherlands), 1955.
3. ——— and A. C. ZAANEN, Notes on Banach function spaces, Proc. Acad. Sci. Amsterdam: Note I, A **66**, 135–147 (1963); Note II, A **66**, 148–153; Note III, A **66**, 239–250; Note IV, A **66**, 251–263; Note V, A **66**, 489–504; Note X, A **67**, 493–506.
4. REICHARD, R., Monotone Norms and Seminorms on an Archimedean Vector Lattice, Thesis, Michigan State University, August, 1970.
5. SOLOVEV, V. A., The extension of a semi-continuous norm and a monotonically complete norm from a KN-lineal to its K-completion, Vestnik Leningrad Univ., 23, 52–57 (1968).
6. VULICH, B. Z., Introduction to the Theory of Partially Ordered Spaces, Wolters-Noordhoff, Groningen, Netherlands, 1967.